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Cantor type attractors in stochastic growth models

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Abstract

We study a one-sector stochastic optimal growth model where production is affected by a shock taking one of two values. Such exogenous shock may enter multiplicatively or additively. A result is presented which provides sufficient conditions to ensure that the attractor of the iterated function system (IFS) representing the optimal policy, is a generalized topological Cantor set. To indicate the role of the strict monotonicity condition on the IFS in this result, examples of attractors, which are not of the Cantor type, are constructed with iterated function systems, whose maps are contractions and satisfy a no overlap property.

1. Introduction

In this paper we provide a further generalization of the framework introduced by Mitra and Privileggi [11], where a stochastic one-sector discounted optimal growth model with an iso-elastic utility function, and a Cobb–Douglas production function affected by a multiplicative random exogenous shock taking one of two values, was investigated. This, in turn, was an expansion of the specific example thoroughly studied in Mitra et al. [10], where utility was assumed to be logarithmic.

Here, the general setting of Brock and Mirman [3] is considered (see also [9]): both the utility function and the production function are any increasing concave twice differentiable functions satisfying the standard assumptions of neoclassical discounted optimal growth models. Two specifications of the model are considered: the case in which the random shocks affect production multiplicatively, and the case in which random shocks are additive. The assumption of a discrete random variable taking one of two values to describe the uncertainty of the model is maintained as in [11]. In such a setting, suitable sufficient conditions on the parameters of the model under which the invariant distribution is supported on a generalized Cantor set are established.

The paper is organized in two main parts. In the first part, after finding a lower bound for the largest fixed point of the lower map of the Iterated Function System (IFS) generated by the optimal policy, we establish a sufficient condition for the crucial *no overlap property* of the IFS, which in turn is a necessary condition to obtain an attractor of the IFS, that is a stable invariant set of the stochastic process of optimal output, with the features of a generalized topological Cantor set.

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In the second part we study topological properties of the attractor of the IFS describing the optimal dynamics. We first define the *generalized topological Cantor set* (a set which is totally disconnected and contains no isolated points) as the attractor of an IFS with nonlinear maps, as opposed to the well known linear "*middle-α*" *Cantor set* obtained as the limit of iterations of linear maps. Then, we use the general theory of IFS to establish that whenever the no overlap property holds and the maps of the IFS are strictly monotone and contractive, the attractor of the IFS is a generalized topological Cantor set. This result applies directly to the findings of the first part of the paper, thus yielding ranges for the values of the parameters of our stochastic one-sector growth model such that its invariant distribution is supported on a generalized topological Cantor set, provided that the maps of the IFS are contractions.

A section of the second part is devoted to construct counter examples that test robustness of the main result. We focus on the essential role played by strict monotonicity: whenever it is relaxed, while the no overlap property is kept in place and the maps are contractions, it becomes straightforward to construct attractors which contain isolated points or non-trivial intervals, and thus cannot be topological Cantor sets.

The outline of the paper is as follows. Section 2 contains a description and basic properties of the model with the assumptions that hold throughout all the subsequent sections. Section 3 is concerned with the no overlap property of the maps constituting the optimal IFS: sufficient conditions for the no overlap property in terms of the parameters of the model are established, both for the multiplicative shocks and for the additive shocks cases. In Section 4 the notion of topological Cantor set is discussed and the main result, establishing conditions under which such a set is the attractor of the IFS describing the optimal dynamics of our growth model, is presented. Some examples of attractors which are not of the Cantor type are illustrated in Section 4.3. Finally, Section 5 reports some concluding remarks. All proofs are gathered in the Appendix A.

2. Preliminaries

We consider the standard model of optimal growth under uncertainty as presented in [3,9]: the production function $f(x,r)$ depends on the amount of capital x employed and on some exogenous shock r which is a random variable taking one of two values, i.e., $r \in \{r_0, r_1\}$, $r_0 < r_1$, where r_0 occurs with probability $p \in (0,1)$ and r_1 with probability $1-p$, independently through time. We shall study two specifications of the production function: one with multiplicative shocks and one with additive shocks. So, there is a function, $h : \mathbb{R}_+ \to \mathbb{R}_+$, such that $f(x,r) = rh(x)$ in the first case and $f(x,r) = h(x) + r$ in the second, for $(x,r) \in \mathbb{R}_+ \times \{r_0, r_1\}$. Both the production function, h, and the utility function, u, are continuous on $\mathbb{R}_+$, and are C^2 functions on $\mathbb{R}_{++}$ satisfying the following standard assumptions:

$$h(0) = 0, \quad h'(\cdot) > 0, \quad h''(\cdot) < 0, \quad \lim_{x\to 0^+} h'(x) = +\infty, \quad \lim_{x\to +\infty} h'(x) = 0, \tag{1}$$

$$u'(\cdot) > 0, \quad u''(\cdot) < 0, \quad \lim_{x\to 0^+} u'(x) = +\infty. \tag{2}$$

Under (1), there is a unique number $k > 0$ such that $h(k) = k$, $h(x) > k$ for all $0 \leqslant x < k$ and $h(x) < k$ for all $x > k$. Thus, a closed interval of the form $[0, k_{r_1}]$ can be taken as the state space for our model. Thus, the "primitives" of our model are the functions h and u, the values r_0, r_1, the probability p and the discount factor $\delta \in (0,1)$.

One can apply the standard theory of stochastic dynamic programming to obtain an (optimal) value function, $V : \mathbb{R}_+ \to \mathbb{R}_+$ and two (optimal) policy functions, $g : \mathbb{R}_+ \to \mathbb{R}_+$ and $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$, which we will interpret as the consumption and the investment functions respectively. That is, given any output level, $y \geqslant 0$, the optimal consumption out of this output is given by $g(y)$, while the optimal input choice (for production in the next period) is then $\gamma(y) = y - g(y)$. In both specifications for the exogenous shocks (multiplicative and additive), we denote $f(\gamma(y), r_0)$ by $G_0(y)$, which gives the output obtained in the next period when r takes the value r_0, and $f(\gamma(y), r_1)$ by $G_1(y)$, which gives the output obtained in the next period when r takes the value r_1. The inverse of h' will play an important role in our analysis, and will be denoted by F.

Following [3,9], one can establish several useful properties of the value and policy functions. We summarize these results (without proofs) in the following Proposition, where we denote $(\partial f/\partial x)(x,r)$ by $f_x(x,r)$.

Proposition 1. *The value function, V, and the policy function, g, satisfy the following properties:*

(i) *V is concave on $\mathbb{R}_+$, and continuous on $\mathbb{R}_{++}$;*
(ii) *g is continuous on $\mathbb{R}_+$ and $0 < g(y) < y$ for $y > 0$;*
(iii) *$g(y)$ and $\gamma(y)$ are both strictly increasing in y on $\mathbb{R}_+$;*
(iv) *for $y > 0$, we have*

$$u'(g(y)) = \delta\{pu'(g(G_0(y)))f_x(\gamma(y), r_0) + (1-p)u'(g(G_1(y)))f_x(\gamma(y), r_1)\}. \tag{3}$$

The optimal policy function leads to the stochastic process:

$$y_{t+1} = \begin{cases} G_0(y_t) & \text{with probability } p \\ G_1(y_t) & \text{with probability } 1-p \end{cases} \quad \text{for } t \geqslant 0. \tag{4}$$

Alternately, one might say that the optimal policy function leads to an iterated function system (IFS) $\{G_0, G_1; p, 1-p\}$. It is known (from [3]), that there is a unique invariant distribution, μ, of the Markov process described by (4), and the distribution of optimal output at date t, call it μ_t, converges weakly to μ.[1] We are principally interested in the geometric properties of the support of μ.

It can be checked that the functions G_0 and G_1 have positive fixed points, and all the fixed points are less than k_{r_1}. Denote by a the largest fixed point of G_0, and by b the smallest fixed point of G_1. Following [3], one can establish that $a < b$. The interval $[a,b]$ is an invariant stable set of the stochastic process (4). In particular, the support of μ is contained in $[a,b]$. Consequently, in studying the support of μ, it is enough to concentrate on the stochastic process (4), with initial output, $y \in [a,b]$. Equivalently, one need only study the IFS $\{G_0, G_1; p, 1-p\}$ on the state space $X = [a,b]$.

3. The no overlap property

Let us examine some elementary features of the IFS $\{G_0, G_1; p, 1-p\}$ on the state space $X = [a,b]$. First, we look at the function G_0. We have $G_0(a) = a$; and, for $y \in (a,b]$, we have $G_0(y) < y$, so the graph of the map lies below the 45^0 line (except at a). Further $G_0(y)$ increases with y, reaching $G_0(b) < G_1(b) = b$ at $y = b$. Next, we look at the function G_1. Clearly, $G_1(a) > G_0(a) = a$; and for all $y \in [a,b)$, we must have $G_1(y) > y$, so the graph of the map lies above the 45^0 line (except at b). Further, $G_1(y)$ increases with y, reaching $G_1(b) = b$ at $y = b$.

We say that the two maps G_0 and G_1 *do not overlap* if:

$$G_0(b) < G_1(a) \tag{5}$$

so that the maximum of the G_0 function is less than the minimum of the G_1 function on the state space $X = [a,b]$.

We want to find conditions on the primitives of the model, specifically, p, δ, r_0, r_1, which ensure the no overlap property (5). We shall obtain similar conditions for the two cases – multiplicative shocks and additive shocks—which are treated separately.

3.1. Multiplicative shocks

Let the production function have the form $f(x,r) = rh(x)$, with h satisfying (1), and let the set of values of the random variable r be $\{r_0, r_1\} = \{q, 1\}$, where $q \in (0,1)$. We interpret the value 1 of r to be the "normal" state, with q representing a downward production shock, occurring with probability $p \in (0,1)$. Therefore, we can re-label the fixed point of h as the number $k_{r_1} = k$ such that $h(k) = k$. The two maps of the IFS are in this case $G_0(y) = qh(\gamma(y))$ and $G_1(y) = h(\gamma(y))$.

We start by establishing a lower bound for the fixed point a of the (lower) map G_0 which depends on the parameters of the model. Recall that F denotes the inverse of h'.

Lemma 1. *The following inequalities hold true*:

$$\gamma(a) > F\left(\frac{1}{\delta pq}\right) \tag{6}$$

and

$$a > qh\left(F\left(\frac{1}{\delta pq}\right)\right). \tag{7}$$

The proof is reported in Appendix A.

[1] For an alternate and simpler approach to this result, see [2].

Remark 1

(i) It is immediately seen that Lemma 1 holds under more general assumptions on the stochastic shocks. In particular, it holds under the assumptions of Lemmas 3.1 and 3.2 in [3]; that is, for any random variable r on some interval $[r_0, r_1]$, with $r_0 > 0$, provided that $\Pr(r_0) > 0$. Moreover it holds for any production function $f(x, r)$ with random shocks that not necessarily enter multiplicatively, but such that $f(x, \cdot)$ is non-decreasing and $f(\cdot, r)$ satisfies conditions similar to (1).

(ii) If, for example, $h(x)$ has the Cobb–Douglas form, that is, $h(x) = x^{1-\alpha}/(1-\alpha)$ for $x \geqslant 0$, where $\alpha \in (0,1)$, then conditions (6) and (7) become $\gamma(a) > [1/(\delta pq)]^{-1/\alpha}$ and $a > [q^{1/\alpha}(\delta p)^{1/\alpha - 1}]/(1-\alpha)$ respectively.

It is convenient to label the lower bound in (7) as follows:

$$\theta_m = qh\left(F\left(\frac{1}{\delta pq}\right)\right). \tag{8}$$

Note that our proof of Lemma 1 shows that θ_m constitutes a lower bound for all fixed points of G_0; specifically, $a > \theta_m$.

Lemma 1 is useful in constructing a sufficient condition for the no overlap property (5) by means of the parameters of the model.

Proposition 2. *Suppose the following condition is satisfied*:

$$\frac{\theta_m}{k} \geqslant q^2, \tag{9}$$

where k is such that $k = h(k)$ and θ_m is defined in (8). *Then the IFS $\{G_0, G_1; p, 1-p\}$ on the state space $X = [a, b]$ has the no overlap property* (5).

The proof is reported in Appendix A.

Remark 2

(i) Note that the no overlap property as stated in (9) does not depend on the utility function u.

(ii) If $h(x)$ has the Cobb–Douglas form, that is, $h(x) = x^{1-\alpha}/(1-\alpha)$ for $x \geqslant 0$, where $\alpha \in (0,1)$, then condition (9) becomes

$$(\delta pq)^{1-\alpha} > [(1-\alpha)kq]^{\alpha}. \tag{10}$$

Since $h(k) = k$, we have $k^{1-\alpha}/(1-\alpha) = k$, that is, $(1-\alpha)^{-1} = k^{\alpha}$. By using this in (10) we easily obtain condition (5) in [11]:

$$q^{2\alpha-1} < [\delta p(1-\alpha)]^{1-\alpha}.$$

3.2. Additive shocks

We turn our attention now to a production function which has the form $f(x, r) = h(x) + r$, with h satisfying (1); moreover, let the set of values of the random variable r be $\{r_0, r_1\} = \{0, q\}$, where $q > 0$. We may interpret the value 0 of r to be the "normal" state, while q represents some positive production shock, occurring with probability $1-p$. The two maps of the IFS are in this case $G_0(y) = h(\gamma(y))$ and $G_1(y) = G_0(y) + q$. Let $\bar{k}$ be the unique fixed point of the map $s(x) = h(x) + q$, so that we have $h(\bar{k}) + q = \bar{k}$. Then, we can set $k_{r_1} = \bar{k}$. Note that $\bar{k} > k + q$, where k is the unique positive fixed point of h. It is also straightforward to show (e.g., by implicit differentiation using condition (1)) that $\bar{k}$ increases as q increases.

A lower bound for the fixed point a of the (lower) map G_0 in this case is defined by the following lemma.

Lemma 2. *The following inequalities hold true*:

$$\gamma(a) > F\left(\frac{1}{\delta p}\right) \tag{11}$$

and

$$a > h\left(F\left(\frac{1}{\delta p}\right)\right). \tag{12}$$

The proof is reported in Appendix A.

Remark 3. Unlike the case where shocks enter production multiplicatively, when the exogenous shock is additive the lower bound for the fixed point a of the (lower) map G_0 does not depend on the shock q itself.

Let us label the lower bound in (7) as follows:

$$\theta_a = h\left(F\left(\frac{1}{\delta p}\right)\right) \tag{13}$$

and state a sufficient condition for the no overlap property (5) to hold for the additive shocks case.

Proposition 3. *Suppose the following condition is satisfied*:

$$\theta_a \geqslant 2h(\bar{k}) - \bar{k}, \tag{14}$$

where $\bar{k}$ *is such that* $\bar{k} = h(\bar{k}) + q$ *and* θ_a *is defined in* (13). *Then the IFS* $\{G_0, G_1; p, 1-p\}$ *on the state space* $X = [a, b]$ *has the no overlap property* (5).

The proof is reported in Appendix A.

Remark 4

(i) Again the no overlap property as stated in (14) does not depend on the utility function u.
(ii) The case where production is affected by an additive shock allows for a more striking interpretation than the previous case with multiplicative shocks. The left term in (14) does not depend on q, while the right term does, since $\bar{k}$ is a strictly increasing function of q; but, under assumption (1), the right term in (14) diverges to $-\infty$ as $\bar{k} \to +\infty$. Therefore, condition (14), and thus the no overlap property (5), holds whenever the shock q is large enough. Note that condition (9) does not allow for a similar interpretation as in that case also the lower bound θ_m does depend on q.

4. Topological structure of the attractor of a IFS

In the previous sections we provided enough information on the IFS $\{G_0, G_1; p, 1-p\}$ defined on the space $X = [a, b]$ so that the standard theory of IFS can be applied (see, e.g., [8,1,4,5]). In view of the examples of Section 4.3, we slightly generalize the setting by considering any pair of continuous maps H_0 and H_1 defined on some compact subset X of the real line; that is, we shall study a generic IFS $\{H_0, H_1; p, 1-p\}$, abstracting from the maps G_0 and G_1 discussed so far.

4.1. A well known result on IFS

Let $X \subset \mathbb{R}$ be a compact set. Let $\mathscr{B}(X)$ denote the sigma-algebra of Borel measurable subsets of X and $\mathscr{P}(X)$ the space of probability measures on $\mathscr{B}(X)$. Recall that the *Barnsley operator* $S: X \to X$ is defined by

$$S(E) = H_0(E) \cup H_1(E) \quad \text{for } E \subseteq X \tag{15}$$

and the *Markov operator* $M : \mathscr{P}(X) \to \mathscr{P}(X)$ is defined by

$$M\mu(B) = p\mu(H_0^{-1}(B)) + (1-p)\mu(H_1^{-1}(B)) \quad \text{for } \mu \in \mathscr{P}(X), \text{ and } B \in \mathscr{B}(X),$$

where $H_0^{-1}(B)$ and $H_1^{-1}(B)$ denote the counter-image sets of the set B through the maps H_0 and H_1 respectively. Operator M describes the evolution of probabilities under the stochastic process

$$y_{t+1} = H_{z_t}(y_t), \tag{16}$$

where z_t are i.i.d. over $\{0, 1\}$ with distribution $\{p, 1-p\}$ for all $t \geqslant 0$. We shall denote the iterates of such operators by $S^t(E) = S(S^{t-1}(E))$ and $M^t(\mu) = M(M^{t-1}(\mu)())$ for all $t \geqslant 1$, with $S^0(E) = E$ and $M^0(\mu) = \mu$.

Recall that the *Hausdorff distance* d_H is defined over the class of all non-empty compact sets in X, $\mathscr{K}(X)$, by

$$d_H(A, B) = \inf\{\delta : A \subset B_\delta \text{ and } B \subset A_\delta\} \quad \text{for } A, B \in \mathscr{K}(X), \tag{17}$$

where A_δ and B_δ denote the *δ-neighborhoods* (*δ-parallel bodies*) of the sets A and B respectively, that is,

$$A_\delta = \{x \in X : |x - a| < \delta \text{ for some } a \in A\}$$

is the set of points within distance δ of A. See, e.g., [4,5] for more details.

In the next proposition are reported (without proof) the main results regarding the attractor and the unique invariant distribution of the IFS $\{H_0, H_1; p, 1-p\}$ on the space $X \subset \mathbb{R}$ induced by the stochastic process (16) when the maps H_0 and H_1 are *contractions*.

Proposition 4. *If constants ℓ_i exist such that $0 < \ell_i < 1$ and $|H_i(y) - H_i(z)| \leqslant \ell_i |y - z|$ for all $y, z \in X$, $i = 0, 1$, then the IFS $\{H_0, H_1; p, 1-p\}$ satisfies the following properties*:

(i) *there is a unique (invariant) compact set $A^* \subseteq X$ such that $S(A^*) = H_0(A^*) \cup H_1(A^*) = A^*$,*
(ii) *for any compact set A_0 such that $S(A_0) \subseteq A_0$, denoting $A_t = S^t(A_0)$ for $t \geqslant 1$, we have $A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots \supseteq A^*$,*
(iii) *A^* is the support of the unique (invariant) probability distribution, $\mu^* \in \mathscr{P}(X)$, satisfying*

$$\mu^*(B) = p\mu^*(H_0^{-1}(B)) + (1-p)\mu^*(H_1^{-1}(B)) \quad \textit{for all } B \in \mathscr{B}(X),$$

(iv) *for $\mu \in \mathscr{P}(X)$, denoting $\mu_t = M^t(\mu)$ for $t \geqslant 1$, μ_t converges weakly to μ^*.*

Proposition 4(ii) states that the iterates of the Barnsley operator, S^t, converge in the Hausdorff distance to the unique set A^*, and that convergence is monotonically decreasing whenever the starting set A_0 is sufficiently large to contain the union of the images of itself through the maps H_0, $H_1 : H_0(A_0) \cup H_1(A_0) \subseteq A_0$. Often, a suitable starting set A_0 to construct a decreasing sequence converging to A^* is the space X itself.

We shall call A^* the *attractor* of the IFS $\{H_0, H_1; p, 1-p\}$ on the space X. For the IFS $\{G_0, G_1; p, 1-p\}$ A^* is thus the support of the invariant distribution μ^* to which the one-sector growth model discussed in the previous sections converges asymptotically.

4.2. Generalized Cantor type attractors

It is well known that if $X = [0, 1]$ and the maps H_0 and H_1 of the IFS are linear with slope m, $0 < m < 1/2$, the attractor A^* of the IFS is a "*middle-α*" *Cantor set*, where $\alpha = 1 - 2m$. This set is obtained by removing the open middle interval of length $0 < \alpha < 1$ from $[0, 1]$ at the first step, then removing the open middle α-proportion from the two disjoint closed intervals remaining after the first step, and continuing the process by removing at each step t the open middle α-proportion from all the 2^t disjoint closed intervals remaining after step $t - 1$, as $t \to +\infty$ (see [10] for a thorough discussion of this example).

The maps of the IFS $\{G_0, G_1; p, 1-p\}$ characterizing the model discussed in the previous sections are clearly nonlinear. The natural question that arises is thus under what conditions such IFS has an attractor that resembles the typical features of a nonlinear Cantor type set. The answer to this question is not obvious as long as nonlinear maps are involved, as it will be illustrated by the examples in Section 4.3.

First we need to make clear what are the main features characterizing a nonlinear Cantor type set. We shall adopt a sufficiently general definition of Cantor set based on topological properties. Recall that a set $E \subseteq X$, where (X, d) is a metric space, is said to be *totally disconnected* if its only connected subsets are one-point sets, that is, for any two distinct points x, y in E, there are two non-empty open disjoint sets U and V such that $x \in U$, $y \in V$ and $(U \cap E) \cup (V \cap E) = E$; also, a set $E \subseteq X$ is said to be *perfect* if it is equal to the set of its accumulation points, that is, it is a closed set which contains no isolated points.

Definition 1. We shall say that a set $C \subset \mathbb{R}$ is a *generalized (topological) Cantor set* on the real line if it is *totally disconnected* and *perfect*.

This definition is fully justified, e.g., in view of Chapter 2 in [7], where it is established that any compact metric space that is totally disconnected and perfect is homeomorphic to the classical "middle-third" Cantor set.

Our objective now is to obtain a set of sufficient conditions on the iterated function system $\{H_0, H_1; p, 1-p\}$ under which the IFS has a unique attractor which is a generalized (topological) Cantor set. This result (stated in Theorem 3 below) can be obtained from the mathematical literature on iterated function systems, and our discussion should be viewed as primarily expository. However, we should note that Theorem 3 is stated in a particularly convenient form for applications (as is clear from our application of it to the optimal growth context in Corollary 1), and the self-contained proof of it (given in Appendix A) is both simple and instructive. The literature on IFS is rather large, but we have not seen a result, exactly in the form of Theorem 3, stated and proved in this literature.

In the economics literature, the attractor of the IFS, generated by the optimal growth model, represents the support of the outputs in a stochastic steady state. Thus, it is important to understand the nature of this attractor. The study of this topic is relatively new, although the existence, uniqueness and stability of the stochastic steady state have been discussed extensively in the literature on optimal growth under uncertainty. A Cantor like attractor is particularly interesting because it suggests that the invariant distribution may have its support on a rather sparse set, while many ranges of intermediate output levels would (almost) never be observed.

Let $\mathrm{diam}(E) = \sup\{|y - z| : y, z \in E\}$ denote the *diameter* of a set $E \subseteq X$. Recall that the *closure* of a set $E \subseteq X$, denoted by $\overline{E}$, is the set containing all accumulation points of E, that is, points that are the limit of some sequence of points in E. We shall denote the composition of maps $f_0 : X \to Y$ and $f_1 : Y \to Z$ by a function $f_0 \circ f_1 : X \to Z$ defined as $(f_0 \circ f_1)(x) = f_0(f_1(x))$; this notation extends to the composition of any finite number of maps in the obvious way. For any $t \geqslant 0$ let us denote a t-sequence of 0s and 1s by $\mathbf{i}_t = (i_0, i_1, \ldots, i_t)$, where $i_k \in \{0, 1\}$ for $k = 0, \ldots, t$, and by Σ_t the set of all such sequences: $\Sigma_t = \{(i_0, i_1, \ldots, i_t) : i_k \in \{0, 1\}, k = 0, \ldots, t\}$. Similarly, let $\mathbf{i}_\infty = (i_0, i_1, \ldots)$ denote an infinite sequence of 0s and 1s, and $\Sigma = \{(i_0, i_1, \ldots) : i_t \in \{0, 1\}, t \geqslant 0\}$ denote the set of all such sequences. With this notation at hand, we can use the shorthand

$$H_{\mathbf{i}_t} = H_{i_0} \circ H_{i_1} \circ \cdots \circ H_{i_t}$$

to denote the composition of the $t + 1$ maps $H_{i_0}, H_{i_1}, \ldots, H_{i_t}$ for a specific sequence of 0s and 1s $\mathbf{i}_t = (i_0, i_1, \ldots, i_t) \in \Sigma_t$.

We shall now see that the set Σ constitutes the natural environment for codifying each element in the attractor A^* (see Chapter IV in [1] for a more exhaustive treatment). Take any compact set $K \subseteq X$ such that $S(K) \subseteq K$; then, by Proposition 4(ii), $A^* = \cap_{t=0}^{\infty} S^t(K)$. On the other hand, by definition of operator S, $S^t(K) = \cup_{\mathbf{i}_t \in \Sigma_t} H_{\mathbf{i}_t}(K)$, and thus

$$A^* = \bigcap_{t=0}^{\infty} \bigcup_{\mathbf{i}_t \in \Sigma_t} H_{\mathbf{i}_t}(K). \tag{18}$$

Note that, since A^* is unique, the right hand side in (18) must be independent of K. By definition of operator S and by Proposition 4(ii), $H_{\mathbf{i}_t}(K) \supseteq H_{\mathbf{i}_{t+1}}(K)$ for all $\mathbf{i}_t \in \Sigma_t$ and $\mathbf{i}_{t+1} \in \Sigma_{t+1}$, hence $H_{\mathbf{i}_t}(K)$ is a decreasing sequence and has a limit as $t \to \infty$. Let $\ell = \max\{\ell_0, \ell_1\}$, then for all $t \geqslant 0$ and for all $\mathbf{i}_t \in \Sigma_t$, $\mathbf{i}_{t+1} \in \Sigma_{t+1}$, $\mathrm{diam}(H_{\mathbf{i}_{t+1}}(K)) \leqslant \ell\, \mathrm{diam}(H_{\mathbf{i}_t}(K)) < \mathrm{diam}(H_{\mathbf{i}_t}(K))$, and thus the diameter of all sets $H_{\mathbf{i}_t}(K)$ vanishes as $t \to \infty$; since the sets $H_{\mathbf{i}_t}(K)$ are compact for all $t \geqslant 0$, the limit of the sequence $H_{\mathbf{i}_t}(K)$ must consist of a single point:

$$y = \bigcap_{t=0}^{\infty} H_{\mathbf{i}_t}(K) \in A^*,$$

which again must be independent of K. Through this construction we can define a map

$$\Pi : \Sigma \to A^* \tag{19}$$

associating with each element of the set Σ (that is, each sequence of 0s and 1s $\mathbf{i}_\infty = (i_0, i_1, \ldots)$), some point of the attractor A^*.

Theorem 1 reports some useful properties of the map (19). For this purpose, we need to introduce a distance ρ for the set Σ so that we can work on a metric space. For any pair of sequences $\mathbf{i}_\infty, \mathbf{j}_\infty \in \Sigma$, let

$$\rho(\mathbf{i}_\infty, \mathbf{j}_\infty) = \ell_{i_0} \ell_{i_1} \cdots \ell_{i_\varphi}, \tag{20}$$

where $i_k \in \{0, 1\}$ for $k = 0, \ldots, \varphi$, and $\varphi = \max\{t : \mathbf{i}_t = \mathbf{j}_t\}$ is the largest t such that the first t elements in the sequences $\mathbf{i}_\infty$ and $\mathbf{j}_\infty$ coincide. If we agree to set $\rho(\mathbf{i}_\infty, \mathbf{j}_\infty) = 1$ when $i_0 \neq j_0$ and $\rho(\mathbf{i}_\infty, \mathbf{j}_\infty) = 0$ if $\mathbf{i}_\infty = \mathbf{j}_\infty$, then it can be easily shown that ρ satisfies the properties of a distance. The metric space (Σ, ρ) is often called *coding space*.

Finally, we generalize property (5) discussed in Section 3 by saying that the maps $H_i : X \to X$, $i = 0, 1$, have the no overlap property[2] if

$$\max_{x \in X} H_0(x) < \min_{x \in X} H_1(x). \tag{21}$$

Theorem 1. *The map $\Pi : \Sigma \to A^*$ defined by*

$$y = \Pi(\mathbf{i}_\infty) = \bigcap_{t=0}^{\infty} H_{\mathbf{i}_t}(K)$$

for some compact set $K \subseteq X$ such that $S(K) \subseteq K$, satisfies the following properties:

[2] Note that the no overlap condition (21) in this context is equivalent to the *strong separation condition* defined on p. 35 in [4].

(i) *it is independent of the set K and is onto,*
(ii) *it is Lipschitz with respect to the distance defined in* (20), *with Lipschitz constant given by* $\mathrm{diam}(A^*)$, *that is,*

$$|\Pi(\mathbf{i}_\infty) - \Pi(\mathbf{j}_\infty)| \leqslant \mathrm{diam}(A^*)\rho(\mathbf{i}_\infty, \mathbf{j}_\infty) \quad \textit{for all } \mathbf{i}_\infty, \mathbf{j}_\infty \in \Sigma,$$

and hence Π *is continuous,*
(iii) *if the maps* H_i, $i = 0, 1$, *are injections and the no overlap property* (21) *holds, then* Π *is bijective.*

Theorem 1 is well known in the literature on fractals; for a full treatment, a good reference is Chapter IV in [1]. Note that, since H_{i_k} are contractions for all $i_k \in \{0,1\}$ and $k = 0, \ldots, t$, $H_{\mathbf{i}_k}$ is also a contraction, and therefore it has a unique fixed point, which will be denoted by $\mathrm{fix}(H_{\mathbf{i}_t})$. The following theorem is due to Williams [13].

Theorem 2. *The unique attractor* A^* *of the IFS* $\{H_0, H_1; p, 1-p\}$ *is the closure of the set of fixed points of arbitrary finite compositions* $H_{\mathbf{i}_t}$, *for all* $t \geqslant 0$, *namely,*

$$A^* = \overline{\bigcup_{t=0}^{\infty} \bigcup_{\mathbf{i}_t \in \Sigma_t} \mathrm{fix}(H_{\mathbf{i}_t})}.$$

See [13] or [8] for general proofs.

Theorem 3. *Suppose that the maps* $H_i : X \to X$, $i = 0, 1$, *are strictly monotone on some closed interval* $X = [a, b]$ *and constants* ℓ_i *exist such that* $0 < \ell_i < 1$ *and* $|H_i(y) - H_i(z)| \leqslant \ell_i |y - z|$ *for all* $y, z \in X$ *and* $i = 0, 1$, *moreover assume that the no overlap property* (21) *holds. Then the unique attractor* A^* *of the IFS* $\{H_0, H_1; p, 1-p\}$ *is totally disconnected and perfect, and therefore it is a generalized (topological) Cantor set.*

A self-contained proof is reported in Appendix A. For a generalization of Theorem 3, see Theorem 3.4 in [6].

The following section contains examples illustrating the role of the assumptions in Theorem 3. All three main assumptions, no overlap, contractivity and strict monotonicity of the maps H_is, seem to be essential. Clearly, the role of no overlap is needed to have "holes" spreading during iterations of operator S^t, a necessary requirement for the attractor to be a Cantor type set. The role of the other two assumptions appears more subtle. Contractivity, besides assuring existence and uniqueness of the attractor A^* as stated in Proposition 4, causes the diameter of the components of each pre-fractal to shrink fast enough so that enough space for the new appearing holes to survive is left after iterates of operator S^t. Strict monotonicity prevents such components to shrink too fast so that the attractor can have neither isolated points nor components which can remain connected.

We conclude this section by applying Theorem 3 to the one-sector growth model discussed in Sections 2 and 3. Note that strict monotonicity of the optimal policy postulated by Proposition 1(iii) implies that the maps G_0 and G_1 of the IFS describing the evolution of optimal output levels through time must be always strictly increasing; thus the only conditions required for the attractor of the model to be a Cantor set are the no overlap property, discussed in Section 3, and contractivity of the maps G_0 and G_1.

Corollary 1. *Assume that the maps* G_0 *and* G_1 *satisfy the no overlap property* (5)*—i.e., either condition* (9) *for the multiplicative shocks case, or condition* (14) *for the additive shocks case—and that constants* ℓ_i *exist such that* $0 < \ell_i < 1$ *and* $|G_i(y) - G_i(z)| \leqslant \ell_i |y - z|$ *for all* $y, z \in X$ *and* $i = 0, 1$. *Then the attractor* A^* *of the IFS* $\{G_0, G_1; p, 1-p\}$ *associated to the stochastic process* (4) *is a generalized (topological) Cantor set.*

The goal of establishing sufficient conditions (on the primitives of the one-sector optimal growth model) for the maps G_0 and G_1 to be contractions directly in terms of the parameter of the growth model is the topic of a companion paper under preparation.

4.3. Examples

The aim of this section is to stress the role of strict monotonicity in Theorem 3. The following examples show that when strict monotonicity is relaxed, the conclusion of Theorem 3 no longer holds. Indeed, under such relaxation, we are able to construct examples of attractors which are either purely isolated points or the union of non-trivial intervals, even while the other assumptions, no overlap and contractivity, are kept in place. Note that in all examples we assume that the maps H_is are non-decreasing, that is, only strict monotonicity (or, more generally, injectiveness), as required by Theorem 3, is dropped. We shall use C^2 maps in order to dispel any doubt that we might be looking for pathological cases. Moreover, if the maps H_is are C^2, it is well known that the IFS $\{H_0, H_1; p, 1-p\}$ can be obtained as the solution of some concave stochastic dynamic programming problem (see [12]).

We shall assume that H_0 and H_1 are contractions on some interval $X = [a, b]$, that is, constants ℓ_i exist such that $0 < \ell_i < 1$ and $|H_i(y) - H_i(z)| \leqslant \ell_i|y - z|$ for all $y, z \in X$ and $i = 0, 1$, and that H_0 and H_1 are only non-decreasing, that is, $H_i(y_1) \leqslant H_i(y_2)$ whenever $y_1 \leqslant y_2$ for $i = 0, 1$. The last assumption allows us to restate the no overlap property, condition (21), as follows:

$$H_0(b) < H_1(a),$$

which will be assumed in all examples.

We start with an extreme example producing a trivial attractor of purely isolated points, followed by a non trivial example again exhibiting an attractor of purely isolated points.

Example 1. Consider the following maps defined on some interval $[a, b]$:

$$H_0(y) \equiv a, \qquad H_1(y) \equiv b.$$

These maps are clearly C^2 and non-decreasing on $X = [a, b]$. $H_0(b) = a < b = H_1(a)$ and thus there is no overlap and also contractivity is trivially satisfied. As it can be seen in Fig. 1(a), the attractor of the IFS $\{H_0, H_1; p, 1 - p\}$ on $X = [a, b]$ is $A^* = \{a, b\}$, which is a set of two isolated points and, clearly, it is not of the Cantor type, as is totally disconnected but not perfect. The set A^* is invariant for the IFS and is produced after the first iteration of the stochastic process (16).

Example 2. Consider the following maps:

$$H_0(y) = \begin{cases} 0 & \text{for } 0 \leqslant y \leqslant 1/4, \\ (72/5)y^3 - (54/5)y^2 + (27/10)y - 9/40 & \text{for } 1/4 \leqslant y \leqslant 1/3, \\ -(36/5)y^3 + (54/5)y^2 - (9/2)y + 23/40 & \text{for } 1/3 \leqslant y \leqslant 2/3, \\ (72/5)y^3 - (162/5)y^2 + (243/10)y - 233/40 & \text{for } 2/3 \leqslant y \leqslant 3/4, \\ 1/4 & \text{for } 3/4 \leqslant y \leqslant 1, \end{cases}$$

$$H_1(y) = \begin{cases} 3/4 & \text{for } 0 \leqslant y \leqslant 1/4, \\ (72/5)y^3 - (54/5)y^2 + (27/10)y + 21/40 & \text{for } 1/4 \leqslant y \leqslant 1/3, \\ -(36/5)y^3 + (54/5)y^2 - (9/2)y + 53/40 & \text{for } 1/3 \leqslant y \leqslant 2/3, \\ (72/5)y^3 - (162/5)y^2 + (243/10)y - 203/40 & \text{for } 2/3 \leqslant y \leqslant 3/4, \\ 1 & \text{for } 3/4 \leqslant y \leqslant 1. \end{cases}$$

It can be shown that these piecewise maps are C^2 and non-decreasing on $X = [0, 1]$. $H_0(1) = 1/4 < 3/4 = H_1(0)$ and thus there is no overlap. Contractivity can be easily checked by computing derivatives on $y = 1/2$, which is the point where both maps are steepest:

$$H_0'(1/2) = H_1'(1/2) = 9/10 < 1$$

and thus they are both contractions. As it can be seen in Fig. 1(b), the attractor of the IFS $\{H_0, H_1; p, 1 - p\}$ on $X = [0, 1]$ is $A^* = \{0, 1/4, 3/4, 1\}$, which is a set of four isolated points and, clearly, it is not of the Cantor type, as is

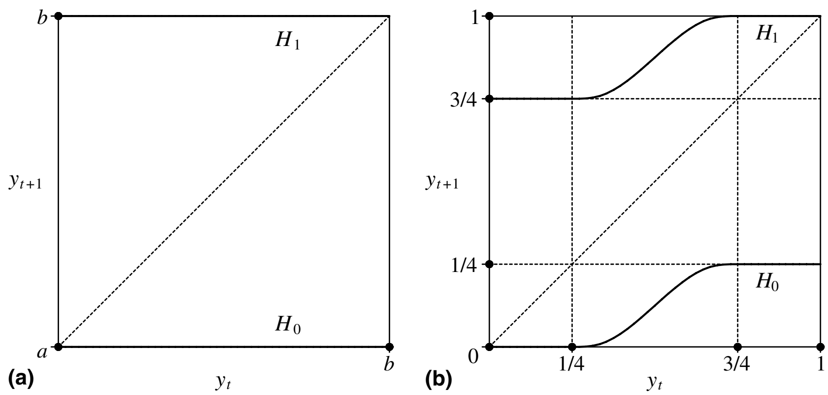

Fig. 1. (a) H_0 and H_1 as in Example 1; (b) H_0 and H_1 as in Example 2.

totally disconnected but not perfect. The set A^* is invariant for the IFS and is produced just after two iterations of the stochastic process (16).

The next example shows how it is possible to construct an IFS with an attractor which is the union of two non-trivial intervals. Such an attractor is definitely a perfect set, but it is not totally disconnected.

Example 3. Consider the following maps:

$$H_0(y) = \begin{cases} (225/8)y^3 & \text{for } 0 \leqslant y \leqslant 1/9, \\ -(75/4)y^3 + (75/8)y^2 - (5/8)y + 1/72 & \text{for } 1/15 \leqslant y \leqslant 4/15, \\ (225/8)y^3 - (225/8)y^2 + (75/8)y - 7/8 & \text{for } 4/15 \leqslant y \leqslant 1/3, \\ 1/6 & \text{for } 1/3 \leqslant y \leqslant 2/3, \\ (225/8)y^3 - (225/8)y^2 + (75/2)y - 49/6 & \text{for } 2/3 \leqslant y \leqslant 11/15, \\ -(75/4)y^3 + (375/8)y^2 - (305/8)y + 743/72 & \text{for } 11/15 \leqslant y \leqslant 14/15, \\ (225/8)y^3 - (675/8)y^2 + (675/8)y - 667/24 & \text{for } 14/15 \leqslant y \leqslant 1, \end{cases}$$

$$H_1(y) = \begin{cases} (225/8)y^3 + 2/3 & \text{for } 0 \leqslant y \leqslant 1/9, \\ -(75/4)y^3 + (75/8)y^2 - (5/8)y + 49/72 & \text{for } 1/15 \leqslant y \leqslant 4/15, \\ (225/8)y^3 - (225/8)y^2 + (75/8)y - 5/24 & \text{for } 4/15 \leqslant y \leqslant 1/3, \\ 1/6 & \text{for } 1/3 \leqslant y \leqslant 2/3, \\ (225/8)y^3 - (225/8)y^2 + (75/2)y - 15/2 & \text{for } 2/3 \leqslant y \leqslant 11/15, \\ -(75/4)y^3 + (375/8)y^2 - (305/8)y + 791/72 & \text{for } 11/15 \leqslant y \leqslant 14/15, \\ (225/8)y^3 - (675/8)y^2 + (675/8)y - 217/8 & \text{for } 14/15 \leqslant y \leqslant 1. \end{cases}$$

It can be checked that these piecewise maps are C^2 and non-decreasing on $X = [0,1]$. $H_0(1) = 1/3 < 2/3 = H_1(0)$ and thus there is no overlap. They are contractions, as their derivatives are bounded by their values on $y = 1/6$:

$$H_0'(1/6) = H_1'(1/6) = 15/16 < 1.$$

Fig. 2 shows that the attractor of the IFS $\{H_0, H_1; p, 1-p\}$ on $X = [0,1]$ is $A^* = [0,1/3] \cup [2/3,1]$, that is, the disjoint union of two closed non-empty intervals. This is not a Cantor type set, as it is perfect but not totally disconnected. The set A^* is invariant for the IFS and is produced just after the first iteration of the stochastic process (16).

These examples show how attractors which are not of the Cantor type can be constructed by relaxing strict monotonicity of the maps H_is: the trick to obtain an attractor of purely isolated points versus an attractor which is the union of closed non-empty intervals is to choose maps which are flat in some appropriate subset of the interval $X = [a,b]$.

Remark 5. It is important to stress that attractors of the kind described in the previous examples, which are not of the Cantor type, are ruled out in the one-sector optimal growth model of Section 2 by Corollary 1. In other words, the main finding of the present work is that whenever the no overlap property holds and the maps representing the optimal policy

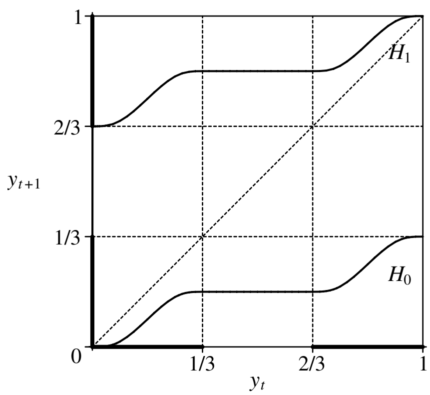

Fig. 2. H_0 and H_1 as in Example 3.

are contractions the attractor of the stochastic one-sector growth model is necessarily a generalized Cantor set (since the optimal policy generates an IFS with strictly increasing maps).

5. Concluding remarks

The main results of this work, Theorem 3 and Corollary 1, provide sufficient conditions on the stochastic one-sector growth model described in Section 2 so that the invariant probability distribution to which the model converges in the long run is supported on a topological Cantor set. Proposition 1(iii) and Propositions 2 and 3, provide conditions on the parameters of the model for two of the three sufficient conditions of Theorem 3 to hold: monotonicity and no overlap property. If, in addition, the maps of the iterated function system (4) are contractions, then Corollary 1 holds. The problem of finding conditions in terms of the parameters of the model, such that the maps describing the optimal policy turn out to be contractions (thus filling the gap left out by the last condition needed to apply Theorem 3), is addressed in ongoing research by the authors, to be reported at a future date.

Appendix A

Proof of Lemma 1. Clearly, (7) follows immediately from (6) by strict monotonicity of h and since a is a fixed point for G_0, that is, $a = G_0(a) = qh(\gamma(a))$.

To prove (6), take any fixed point $\bar{y}$ for the map G_0, $\bar{y} = G_0(\bar{y})$. We calculate the stochastic Ramsey–Euler equation (3) (Proposition 1(iv)) at $y = \bar{y}$:

$$\begin{aligned} u'(g(\bar{y})) &= \delta\{pu'(g(G_0(\bar{y})))qh'(\gamma(\bar{y})) + (1-p)u'(g(G_1(\bar{y})))h'(\gamma(\bar{y}))\} > \delta pu'(g(G_0(\bar{y})))qh'(\gamma(\bar{y})) \\ &= \delta pu'(g(\bar{y}))qh'(\gamma(\bar{y})), \end{aligned}$$

where the last equality holds since $\bar{y} = G_0(\bar{y})$. Thus, we have

$$\frac{1}{\delta pq} > h'(\gamma(\bar{y})).$$

By applying the decreasing function, F, to both sides we get

$$F\left(\frac{1}{\delta pq}\right) < \gamma(\bar{y})$$

and since $\bar{y}$ is an arbitrary fixed point for the map G_0, inequality (6) is established. □

Proof of Proposition 2. Since $G_1(a) = (a/q)$ and $G_0(b) = qG_1(b) = qb$, the no overlap condition (5) is equivalent to

$$qb < \frac{a}{q}. \tag{22}$$

As $b \leqslant k$, a sufficient condition for (22) to hold is $qk < a/q$, which, since $a > \theta_m$, leads immediately to condition (9). □

Proof of Lemma 2. As in Proof of Lemma 1, (12) follows immediately from (11) by strict monotonicity of h and since a is a fixed point for G_0, that is, $a = G_0(a) = h(\gamma(a))$. For any fixed point $\bar{y}$ of the map G_0, $\bar{y} = G_0(\bar{y})$, through a similar use of the stochastic Ramsey–Euler equation (3) as in Proof of Lemma 1, we easily obtain

$$\frac{1}{\delta p} > h'(\gamma(\bar{y})).$$

By applying the decreasing function, F, to both sides we get

$$F\left(\frac{1}{\delta p}\right) < \gamma(\bar{y})$$

and since $\bar{y}$ is an arbitrary fixed point for the map G_0, inequality (11) is established. □

Proof of Proposition 3. Since $G_0(b) = G_1(b) - q = b - q$ and $G_1(a) = a + q$, the no overlap condition (5) is equivalent to

$$b - a < 2q. \tag{23}$$

As $b \leqslant \bar{k}$ and $a > \theta_a$, a sufficient condition for (23) is $\bar{k} - \theta_a \leqslant 2q$, which, by substituting $q = \bar{k} - h(\bar{k})$, yields (14). □

Proof of Theorem 3. Since $S(X) \subseteq X$, we can use Proposition 4(ii) to construct a monotonically decreasing sequence of sets converging to A^* in the Hausdorff distance starting from $X = [a, b]$: denoting $A_t = S^t(X)$ for $t \geqslant 0$, we have $X = A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots \supseteq A^*$. The sets A_t are called *pre-fractals*, as they provide increasingly better estimations of the attractor A^* as t becomes larger. Note that, if the starting set is $A_0 = X = [a, b]$, all pre-fractals A_t are the union of closed intervals, which are called *components* of the pre-fractal A_t. Clearly each component is a set of the type $H_{\mathbf{i}_t}(X)$ for some sequence $\mathbf{i}_t \in \Sigma_t$. Since the maps H_i are strictly monotone—and thus they are injections—and the no overlap property (21) holds, for all $t \geqslant 0$ the pre-fractal A_t is the union of 2^t non-empty closed disjoint intervals: $A_t = \cup_{\mathbf{i}_t} \in \Sigma_t H_{\ \mathbf{i}_t}(X)$ with $H_{\mathbf{i}_t}(X) \cap H_{\mathbf{j}_t}(X) = \emptyset$ for $\mathbf{i}_t \neq \mathbf{j}_t$. Moreover Theorem 1(iii) applies and, for any two points $y, z \in A^*$ such that $y \neq z$, we can write $y = \Pi(\mathbf{i}_\infty) = \cap_{t=0}^{\infty} H_{\mathbf{i}_t}(X)$ and $z = \Pi(\mathbf{j}_\infty) = \cap_{t=0}^{\infty} H_{\mathbf{j}_t}(X)$ with $\mathbf{i}_\infty, \mathbf{j}_\infty \in \Sigma$ and $\mathbf{i}_\infty \neq j_\infty$. But this implies that there is $0 \leqslant t < \infty$ such that $\mathbf{i}_t \neq \mathbf{j}_t$ with $y \in H_{\mathbf{i}_t}(X)$ and $z \in H_{\mathbf{j}_t}(X)$. Since $H_{\mathbf{i}_t}(X)$ and $H_{\mathbf{j}_t}(X)$ are closed and disjoint, $A^* \subseteq A_t$, and y, z are arbitrary, this is enough to establish that A^* is totally disconnected.

To show that A^* is also perfect we shall use Theorem 2. We must show that every point $y \in A^*$ is the limit of some sequence of (distinct) points in A^*. Let $y \in A^*$; then, by Theorem 2, either (a) $y = \text{fix}(H_{\mathbf{i}_t})$ for some $\mathbf{i}_t \in \Sigma_t$, $t \geqslant 0$, or (b) it is the limit of some sequence of such points, $y = \lim_{k\to\infty} y_k$ where, for all k, $y_k = \text{fix}(H_{\mathbf{i}_t})$ for some $i_t \in \Sigma_t$, $t \geqslant 0$. Let us consider case (a) and assume that $y = \text{fix}(H_{\mathbf{i}_t})$ for some $\mathbf{i}_t \in \Sigma_t$, $t \geqslant 0$; that is, $y = H_{\mathbf{i}_t}(y)$. Now choose $i \in \{0, 1\}$ so that $z = \text{fix}(H_i)$ and $z \neq y$; since there are two distinct maps H_0 and H_1 in the IFS, such choice is always possible. Clearly, by Theorem 2, $z \in A^*$. Define the sequence $y_k = (H_{\mathbf{i}_t})^k(z)$, where $(H_{\mathbf{i}_t})^k = H_{\mathbf{i}_t} \circ \cdots \circ H_{\mathbf{i}_t}$ denotes the k-fold composition of the map $H_{\mathbf{i}_t}$. As $H_{\mathbf{i}_t}$ maps A^* into itself and $z \in A^*$, $y_k \in A^*$ for all k. Since $H_{\mathbf{i}_t}$ is a contraction and is strictly monotone, so is $(H_{\mathbf{i}_t})^k$, and thus the sequence y_k constructed so far converges to y and contains distinct elements in A^* for all k; hence y is an accumulation point of A^*. As far as case (b) is considered, note that in this case $y_k \in A^*$ for all k; thus y turns out to be an accumulation point of A^* by definition, and the proof is complete. □

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